# Fine structure generation in a double-diffusive system

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Double-diffusive convection in a horizontally infinite layer of a unit height in a large-Rayleigh-number limit is considered. From linear stability analysis it is shown that the convection tends to have a form of traveling tall thin rolls with width about 30 times less than height. Amplitude equations of *ABC* type for vertical variations of the amplitude of these rolls and mean values of diffusive components are derived. As a result of its numerical simulation it is shown that for a wide variety of parameters considered *ABC* system have solutions, known as diffusive chaos, which can be useful for the explanation of fine structure generation in some important oceanographical systems like thermohaline staircases.

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## I. INTRODUCTION

Double-diffusive or thermohaline convection plays an important role in heat-mass transfer processes in the ocean [1]. It also essentially influences various small-scale processes, like the formation of vertical temperature and salinity fine structure. Such phenomena are not well understood at the present day. There are only a few works devoted to analytical models of fine structure generation in the sea [2–4]. But none of them essentially considers the role of double diffusion in such processes. As an exception we can mention Ref. [5], where double-diffusive steplike fine structure is simulated by numerical Monte Carlo methods.

The purpose of this paper is to develop a mathematical model of two-dimensional double-diffusive convection in a horizontally infinite layer, based on a system of amplitude equations, which describe the formation of a vertical fine structure, which in some aspects resembles actual experimental and observational data. The main idea of this work consists in the result that, in the limit of large Rayleigh numbers, convective cells tend to be narrow and tall, which gives an opportunity to construct an *ABC* system of amplitude equations [6] with respect to the vertical coordinate as an evolution variable in the case of such cells. This paper includes two main sections besides this one. In Sec. II we determine the sizes of the most prominent cells in the large-Rayleigh-number limit by means of linear stability analysis. In Sec. III we derive the *ABC* system of amplitude equations by the multiple-scale method. The numerical simulation provides solutions for vertical fine structure, which have features of so-called diffusive chaos.

We consider two-dimensional thermohaline convection in a fluid layer of thickness $h$, bounded by two horizontal infinite planes. In the Cartesian frame with the horizontal $x$ axis and $z$ axis directed upward the motion is described by the stream function $\psi(t,x,z)$, where $t$ is the time variable. It is assumed that there are no distributed sources of heat and salt, and on the upper and lower boundaries of the layer these quantities have constant values. Hence, the basic distribution of temperature and salinity is linear along the vertical direction and does not depend on time. The variables $\theta(t,x,z)$ and $\xi(t,x,z)$ describe variations in the temperature and salinity about this main distribution. There are two types of thermohaline convection: fingering [7], in which the warmer and more saline liquid is at the upper boundary of the area, and diffusive type, in which the temperature and salinity are greater at the lower boundary [8]. In this paper we study the latter case.

The dimensionless governing equations in the Boussinesq approximation for the momentum and diffusion of temperature and salt are [9,10]

$$(\partial_t - \sigma\Delta)\Delta\psi + \sigma(R_S\partial_x\xi - R_T\partial_x\theta) = J(\Delta\psi,\psi),$$

$$(\partial_t - \Delta)\theta - \partial_x\psi = J(\theta,\psi), \tag{1}$$

$$(\partial_t - \tau\Delta)\xi - \partial_x\psi = J(\xi,\psi),$$

where $J(f,g)=\partial_x f\partial_z g - \partial_x g\partial_z f$ is the Jacobian, $\sigma$ is the Prandtl number (usual value is 7.0), $\tau$ is the Lewis number ($0<\tau<1$, usually $0.01-0.1$), and $R_T$ and $R_S$ are the temperature and salinity Rayleigh numbers.

The boundary conditions for the dependent variables are chosen to be zero, which implies that the temperature and salinity at the boundaries of the area are constants, the vorticity vanishes at the boundaries, and the boundaries are impermeable:

$$\psi = \partial_z^2\psi = \theta = \xi = 0 \quad \text{on } z=0,\ 1. \tag{2}$$

These boundary conditions are usually called free-slip conditions because the horizontal velocity component at the boundary is not specified.

As a space scale the thickness of the liquid layer, $h$, is used. As a time scale the value $t_0=h^2/\chi$ is used, where $\chi$ is the thermal diffusivity of the liquid. Velocity field components are determined as $v_z=(\chi/h)\partial_x\psi$ and $v_x=-(\chi/h)\partial_z\psi$. For temperature $T$ and salinity $S$ we have the expressions

$$T(t,x,z) = T_- + (T_+ - T_-)[1 - z + \theta(t,x,z)],$$

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$$S(t,x,z) = S_- + (S_+ - S_-)[1 - z + \xi(t,x,z)],$$

where $T_+, T_-$ and $S_+, S_-$ are the temperatures and salinities on the lower and upper boundaries of the layer, respectively. The temperature and salinity Rayleigh numbers can be expressed as follows:

$$R_T = \frac{g\alpha' h^3}{\chi\nu}(T_+ - T_-), \quad R_S = \frac{g\gamma' h^3}{\chi\nu}(S_+ - S_-),$$

where $g$ is the acceleration of gravity, $\nu$ is the kinematic viscosity of fluid, and $\alpha'$ and $\gamma'$ are the temperature and salinity coefficients of volume expansions.

## II. FORM OF CONVECTIVE CELLS AT LARGE RAYLEIGH NUMBERS

Consider thermohaline convection at large $R_S$, which is appropriate for the most of oceanographically important applications ($R_S \approx 10^9 - 10^{12}$). After rescaling the time $t=(\sigma R_S)^{-1/2}t'$ and the stream function $\psi=(\sigma R_S)^{1/2}\psi'$, we can rewrite the basic system (1) in a singularly disturbed form (primes are omitted)

$$(\partial_t - \sigma\varepsilon^2\Delta)\Delta\psi + [\partial_x\xi - (1-N^2)\partial_x\theta] = J(\Delta\psi,\psi),$$

$$(\partial_t - \varepsilon^2\Delta)\theta - \partial_x\psi = J(\theta,\psi), \tag{3}$$

$$(\partial_t - \tau\varepsilon^2\Delta)\xi - \partial_x\psi = J(\xi,\psi).$$

Here a small parameter $\varepsilon = \sqrt[4]{1/\sigma R_S}$ and a buoyancy frequency $N=\sqrt{1-R_T/R_S}$ are introduced. If we put $\varepsilon=0$, then our system (3) turns into equations describing the two-dimensional internal waves with the constant buoyancy frequency $N$ in the Boussinesq approximation.

In order to investigate the linear stability problem for the system (3) with boundary conditions (2) we omit nonlinear terms in the right part of the system and choose a normal mode solution

$$\psi(x,z,t) = Ae^{\lambda t - ikx}\sin n\pi z,$$

$$\theta(x,z,t) = a_T e^{\lambda t - ikx}\sin n\pi z, \tag{4}$$

$$\xi(x,z,t) = a_S e^{\lambda t - ikx}\sin n\pi z,$$

where $\lambda$ describes the growth rate of the mode, $k$ is a horizontal wave number, $n$ is a number of the mode, and $A$ is an amplitude of the mode. After substitution of expressions (4) into the system (3) we get a system of algebraic equations with solvability condition having the form of a cubic equation with respect to $\lambda$:

$$(\lambda + \sigma\varepsilon^2\varkappa^2)(\lambda + \varepsilon^2\varkappa^2)(\lambda + \tau\varepsilon^2\varkappa^2) + \frac{k^2N^2}{\varkappa^2}(\lambda + \gamma\varepsilon^2\varkappa^2) = 0. \tag{5}$$

Here $\varkappa^2=k^2+n^2\pi^2$ is the full wave number and $\gamma$ is a constant: $\gamma=\tau+(1-\tau)/N^2$. Equation (5) is known as a dispersive relation and has three roots, two of which can be complex conjugate for a sufficiently small value of $\varepsilon$. In the latter case the Hopf bifurcation takes place, when for some values of $N$ and $\varepsilon$ the real part of the complex-conjugate roots turns out to be zero. It is true when

$$\varepsilon^4 < \frac{k^2}{\tau^2\varkappa^6}\left(\frac{1-\tau}{1+\sigma}\right),$$

$$N_*^2 = \frac{1-\tau}{1+\sigma} - \varepsilon^4\frac{\varkappa^6}{k^2}[\sigma + \tau(1+\tau+\sigma)],$$

$$\omega^2 = \frac{k^2}{\varkappa^2}\left(\frac{1-\tau}{1+\sigma}\right) - \varepsilon^4\tau^2\varkappa^4,$$

where $\omega=\mathrm{Im}(\lambda)$ is the Hopf frequency.

Because dispersive relation (5) explicitly contains the small parameter, we can choose one of the complex-conjugate roots and express $\lambda$ in the form of an asymptotic expansion in powers of $\varepsilon$:

$$\lambda = \lambda_0 + \varepsilon^2\lambda_1 + \varepsilon^4\lambda_2 + \varepsilon^6\lambda_3 + \cdots .$$

After substitution of this expression into Eq. (5) we have the following expressions for $\lambda_i$:

$$\lambda_0^2 = -(k^2/\varkappa^2)N^2, \quad \lambda_1 = \varkappa^2F_1,$$

$$\lambda_2 = -(\varkappa^4/\lambda_0)F_2, \quad \lambda_3 = -\varkappa^8/(k^2N^2)F_3,$$

$$F_1 = (\gamma - C_1)/2 > 0,$$

$$F_2 = (3F_1^2 + 2C_1F_1 + C_2)/2 > 0,$$

$$F_3 = 4F_1^3 + 4C_1F_1^2 + (C_1^2 + C_2)F_1 + (\tau+\sigma)(C_1 + \tau\sigma)/2 > 0,$$

where $C_1=1+\tau+\sigma$ and $C_2=\tau+\sigma+\tau\sigma$. The growth rate caused by thermohaline convective instability and wave frequency can be written as follows [11]:

$$\mathrm{Re}(\lambda) = \varepsilon^2\varkappa^2F_1 - \varepsilon^6\frac{\varkappa^8}{k^2N^2}F_3 + \cdots ,$$

$$\mathrm{Im}(\lambda) = N\frac{k}{\varkappa}\left(1 + \varepsilon^4\frac{\varkappa^6}{k^2N^2}F_2 + \cdots\right). \tag{6}$$

One can see (Fig. 1) that for a given mode with number $n$ the growth rate has a maximum for some $k$, which determines the horizontal size of the most prominent convective cells. As the horizontal size of convective cell is equal to the horizontal wave period $2\pi/k$, the graph in Fig. 2 shows that the sizes of the most prominent cells are rather narrow, because $k$ of the most rapidly growing cells is about 100—i.e., relatively large.

For simplicity we further consider only the first convective mode.

Rewriting Eq. (5) in another form and introducing the new variables $P^2=\varepsilon^2\varkappa^2 \approx \varepsilon^2k^2$, $X=\lambda/P^2$, and $Y=N^2/P^4$,

$$(X+\sigma)(X+1)(X+\tau) + Y(X+\gamma) = 0.$$

The roots of this equation depend on parameter $Y$, so that finally $\lambda$ depends on the horizontal wave number $P$.

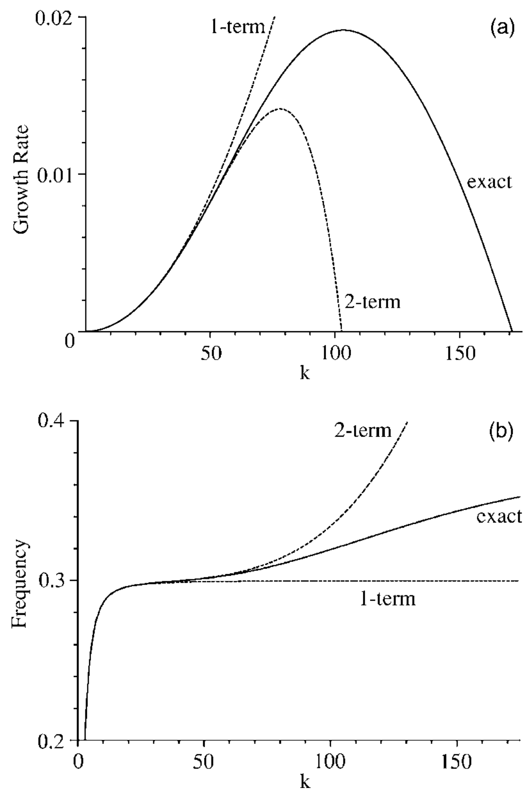


FIG. 1. Growth rate $\mathrm{Re}[\lambda(k)]$ (a) and frequency $\mathrm{Im}[\lambda(k)]$ (b) of traveling waves (4) for the first convective mode. Here $\varepsilon=0.00153$, $\sigma=7$, $\tau=1/81$, and $N=0.3$. Curves labeled as "1-term" are single-term approximations of Eqs. (6); curves labeled as "2-term" are two-term approximations; curves labeled as "exact" are exact solutions of Eq. (5).

For example, consider an actual oceanographical system such as an inversion of the thermohaline staircase. Let it have thickness $h=250$ cm, temperature difference $T_+-T_-=0.1$ °C, $\sigma=7$, and $\tau=1/81$. In this case $\varepsilon=1.53\times10^{-3}$ and the nondimensional critical buoyancy frequency $N_*=0.351\,36$. For $N=0.2764$ the most unstable mode has $P_*=0.159\,957\,9$ and width of convective cell, $l_c=\pi\varepsilon h/P_*\approx7.7$ cm. In comparison with this formula (6) gives $P_*=0.126$, which is somewhat less than the exact value.

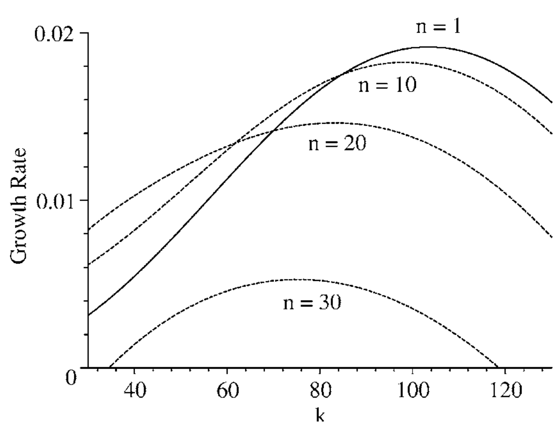


FIG. 2. Growth rate $\mathrm{Re}[\lambda(k,n)]$ for convective modes with different mode number $n$. All parameters are the same as in the Fig. 1.

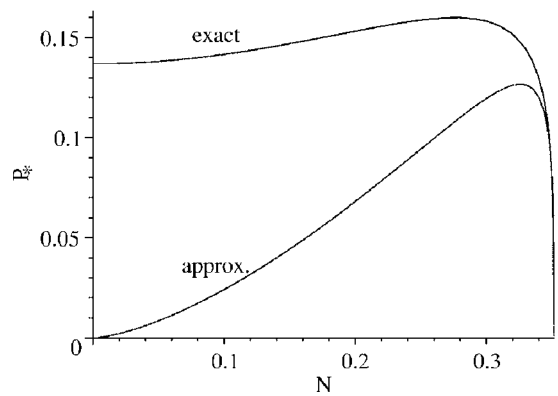


FIG. 3. Dependence of the wave number $P_*=k\varepsilon$ of the most unstable mode from buoyancy frequency $N$. The curve labeled as "exact" is the exact numerical solution; the curve labeled as "approx." is the approximation by formula (6).

From (6) one can extract the dependence of $P_*$ from $N$, in the form $P_*=[N^2F_1/(3F_3)]^{1/4}$.

From the picture shown in Fig. 3 it is easily seen that when the value of $N$ becomes slightly smaller than its critical value $N_*\approx0.351\,36$, the value of $P_*$ abruptly [as $(N_*-N)^{1/4}$] increases and becomes maximal at $N\approx0.3$. When $N$ becomes even smaller, $P_*$ decreases to $P_*\approx0.136\,88$ for $N=0$. It should be emphasized that $P_*$ is nearly independent of $N$ in the considered case.

This result shows that thermohaline convection at large $R_S$ is essentially different from that for small $R_S$, when the critical wave number is $k_*=\pi/\sqrt{2}$ [12]. For our case the typical wave numbers are of order $0.1/\varepsilon$. This estimate is more accurate than the one mentioned in [1]. Thus, according to linear stability analysis, convective cells for large $R_S$ have tall and thin geometry. The more physical explanation of this effect consists in the prevailing of the buoyancy forces acting vertically in comparison with the forces of inertia of the fluid-particle determining width of cells, when Rayleigh numbers are large.

## III. FINE STRUCTURE GENERATION AND DIFFUSIVE CHAOS

In this section we derive an *ABC* system for weak nonlinear vertical modulations of the convective cell amplitude and mean profiles of temperature and salt, and present some numerical results.

First, we introduce a new small parameter, extracted from the geometry of the convective cells: $\delta=l_c/h=\pi\varepsilon/P_*\approx20\varepsilon$. Then we rescale the variables $\psi=\delta^2\psi'$, $\theta=\delta\theta'$, and $\xi=\delta\xi'$ (the prime will be omitted) and introduce one more parameter $E=\varepsilon/\delta\approx1/20$. After changing of the space scale from $h$ to $l_c$, the basic system (3) appears in the form

$$(\partial_t-\sigma E^2\Delta)\Delta\psi+[\partial_x\xi-(1-N^2)\partial_x\theta]=J(\Delta\psi,\psi),$$

$$(\partial_t-E^2\Delta)\theta-\partial_x\psi=J(\theta,\psi), \tag{7}$$

$$(\partial_t - \tau E^2 \Delta)\xi - \partial_x \psi = J(\xi,\psi).$$

Now, we introduce a slow vertical variable $Z=\delta z$ and a slow time variable $T=\delta^2 t$. In accordance with multiple-scale method [13–15] the dependent variables now depend on $t$, $T$, $x$, $z$, and $Z$ which are considered as independent. So we should replace the derivatives in Eqs. (7) with the so-called the prolonged derivatives by the rules

$$\partial_z \rightarrow \delta\partial_Z,$$

$$\partial_t \rightarrow \partial_t + \delta^2 \partial_T,$$

$$\Delta \rightarrow \partial_x^2 + \delta^2 \partial_Z^2,$$

$$\Delta^2 \rightarrow \partial_x^4 + 2\delta^2 \partial_x^2 \partial_Z^2 + \delta^4 \partial_Z^4,$$

$$\partial_t \Delta \rightarrow \partial_t \partial_x^2 + \delta^2 \partial_T \partial_x^2 + \delta^2 \partial_t \partial_Z^2.$$

Let the buoyancy frequency be slightly less than its critical value $N^2 = N_*^2 - \delta^2 R$, where the parameter $R$ describes the forcing of the system. Equations (7) now turn into

$$\begin{aligned}(\partial_t - \sigma E^2 \partial_x^2)\partial_x^2 \psi + [\partial_x \xi - (1-N_*^2)\partial_x \theta] &= -\delta J_Z(\psi, \partial_x^2 \psi) \\ &- \delta^2[(\partial_T \partial_x^2 + \partial_t \partial_Z^2 - 2\sigma E^2 \partial_x^2 \partial_Z^2)\psi - R\partial_x \theta],\end{aligned}$$

$$(\partial_t - E^2 \partial_x^2)\theta - \partial_x \psi = -\delta J_Z(\psi,\theta) - \delta^2(\partial_T - E^2 \partial_Z^2)\theta, \quad (8)$$

$$(\partial_t - \tau E^2 \partial_x^2)\xi - \partial_x \psi = -\delta J_Z(\psi,\xi) - \delta^2(\partial_T - \tau E^2 \partial_Z^2)\xi.$$

We seek the solutions of these equations as the asymptotic series in powers of the small parameter $\delta$:

$$\psi = \delta\psi_1 + \delta^2 \psi_2 + \delta^3 \psi_3 + \cdots,$$

$$\theta = \delta\theta_1 + \delta^2 \theta_2 + \delta^3 \theta_3 + \cdots, \quad (9)$$

$$\xi = \delta\xi_1 + \delta^2 \xi_2 + \delta^3 \xi_3 + \cdots.$$

After substituting these expressions into Eqs. (8) and collecting terms of like powers of $\delta$, we obtain systems of equations for determining of the terms of the series (9). At $O(\delta^1)$ we have the following system:

$$(\partial_t - \sigma E^2 \partial_x^2)\partial_x^2 \psi_1 + [\partial_x \xi_1 - (1-N_*^2)\partial_x \theta_1] = 0,$$

$$(\partial_t - E^2 \partial_x^2)\theta_1 - \partial_x \psi_1 = 0, \quad (10)$$

$$(\partial_t - \tau E^2 \partial_x^2)\xi_1 - \partial_x \psi_1 = 0.$$

We choose for this system a solution in the form of normal convective mode traveling to the right, with constants of integration $B(T,Z)$ and $C(T,Z)$, depending on slow variables:

$$\psi_1 = A(T,Z)e^{i\omega t - iKx} + \text{c.c.},$$

$$\theta_1 = a_T(T,Z)e^{i\omega t - iKx} + B(T,Z) + \text{c.c.}, \quad (11)$$

$$\xi_1 = a_S(T,Z)e^{i\omega t - iKx} + C(T,Z) + \text{c.c.}$$

Here c.c. represents complex conjugate, the wave number $K = K_*(N)$ is a horizontal wave number corresponding to the most unstable waves of convection, and the maximal value of $K$ is $\pi$ from the choice of the space scale related to convective cells. It is attained when $N \approx 0.3$. Parameters of the normal mode (11) can be represented as follows:

$$a_T = -\frac{iK}{i\omega + E^2 K^2}A, \quad a_S = -\frac{iK}{i\omega + \tau E^2 K^2}A,$$

$$(i\omega + \sigma E^2 K^2)(i\omega + E^2 K^2)(i\omega + \tau E^2 K^2) + N^2(i\omega + \gamma E^2 K^2) = 0.$$

The last formula is actually the dispersive relation (5), but for an infinitesimal vertical wave number. Also we can represent critical buoyancy frequency $N_*$ and wave frequency $\omega$ as

$$N_*^2 = \frac{1-\tau}{1+\sigma} - (1+\tau)(\tau+\sigma)E^4 K^4,$$

$$\omega^2 = \frac{1-\tau}{1+\sigma} - \tau^2 E^4 K^4,$$

$$\omega^2 = N_*^2 + (\sigma + \tau + \sigma\tau)E^4 K^4.$$

The system of equations at $O(\delta^2)$ by form is the same as (10) and does not lead to any new results. The system at $O(\delta^3)$ is

$$\begin{aligned}(\partial_t - \sigma E^2 \partial_x^2)\partial_x^2 \psi_3 + [\partial_x \xi_3 - (1-N_*^2)\partial_x \theta_3] &= -J_Z(\psi_1, \partial_x^2 \psi_1) \\ &- (\partial_T \partial_x^2 + \partial_t \partial_Z^2 - 2\sigma E^2 \partial_x^2 \partial_Z^2)\psi_1 + R\partial_x \theta_1,\end{aligned}$$

$$(\partial_t - E^2 \partial_x^2)\theta_3 - \partial_x \psi_3 = -J_Z(\psi_1,\theta_1) - (\partial_T - E^2 \partial_Z^2)\theta_1,$$

$$(\partial_t - \tau E^2 \partial_x^2)\xi_3 - \partial_x \psi_3 = -J_Z(\psi_1,\xi_1) - (\partial_T - \tau E^2 \partial_Z^2)\xi_1.$$

After substitution into the right parts of these equations the expressions of dependent variables from (11) we get a system with resonating right parts breaking regularity of the asymptotic expansions (9). The condition which must be satisfied for there to be no secular terms in this case takes form of the so-called $ABC$ system [6] (intermediate calculations are omitted):

$$\partial_T A = E^2 \beta_1 \partial_Z^2 A + R\beta_2 A - \beta_3 A \partial_Z B + \beta_4 A \partial_Z C,$$

$$\partial_T B = E^2 \partial_Z^2 B - E^2 \beta_5 \partial_Z |A|^2, \quad (12)$$

$$\partial_T C = \tau E^2 \partial_Z^2 C - \tau E^2 \beta_6 \partial_Z |A|^2.$$

The coefficients of these equations are given by

$$\beta_0 = 1 + \frac{1}{i\omega + E^2 K^2}\left[(i\omega + \sigma E^2 K^2) - \frac{(1-\tau)E^2 K^2}{(i\omega + \tau E^2 K^2)^2}\right],$$

$$\begin{aligned}\beta_1 = \Bigg\{\left(\frac{i\omega}{E^2 K^2} + 2\sigma\right) &+ \frac{1}{i\omega + E^2 K^2} \\ &\times \left[(i\omega + \sigma E^2 K^2) + \frac{(1-\tau)i\omega}{(i\omega + \tau E^2 K^2)^2}\right]\Bigg\}\beta_0^{-1},\end{aligned}$$

$$\beta_3 = \left[(i\omega + \sigma E^2 K^2) + \frac{1}{i\omega + \tau E^2 K^2}\right]\beta_0^{-1},$$

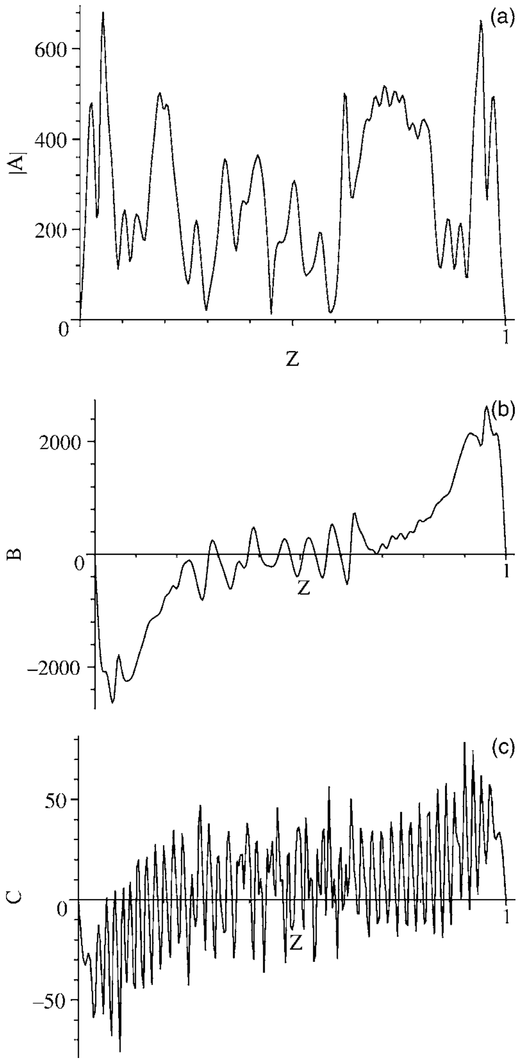


FIG. 4. Numerical solution of the system (13) at the time $t=9.28$ h. Here the variables $|A(T,Z)|$ (a), $B(T,Z)$ (b), and $C(T,Z)$ (c). The forcing parameter is $R=16$, also $\varepsilon=0.00153$, $\sigma=7$, and $\tau=1/81$. Layer depth $h=250$ cm; the number of grid points is 2048. Dimensional variations in temperature and salinity are proportional to $B$ and $C$; it can be estimated as $2.14\times10^{-6}B(T_+-T_-)$ °C and $1.88\times10^{-6}C(S_+-S_-)\%$, respectively. The amplitude of the stream function is proportional to $|A|$ and can be estimated as $2.14\times10^{-5}|A|$ cm$^2$/sec.

$$\beta_2=\frac{\beta_0^{-1}}{i\omega+E^2K^2},\quad \beta_4=\frac{\beta_0^{-1}}{i\omega+\tau E^2K^2},$$

$$\beta_5=\frac{2K^4}{\omega^2+E^4K^4},\quad \beta_6=\frac{2K^4}{\omega^2+\tau^2E^4K^4}.$$

For the aims of numerical simulation, transform system (12) to a more convenient form by introducing a new time variable $T'=E^2T$ and applying the following substitutions:

$$A'=AE^{-1}\sqrt{\beta_5|\beta_3|}\exp(-iR\beta_{2R}T'E^{-2}),$$

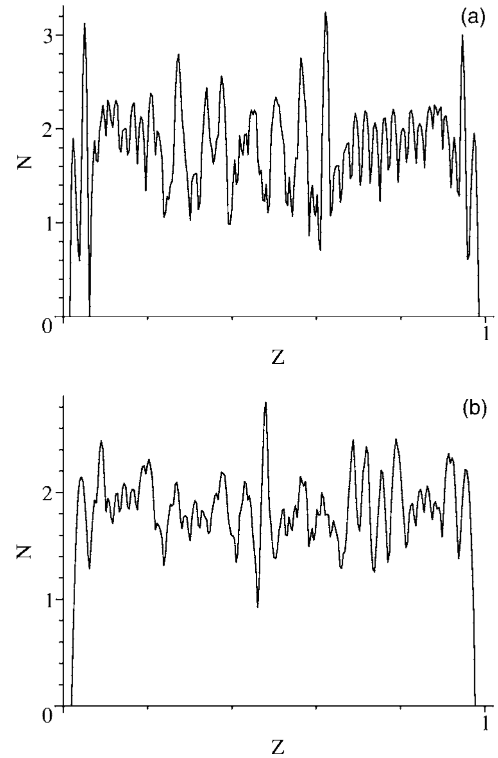


FIG. 5. Buoyancy frequency $N$ (cycles per hour) vertical microstructure for time $t=9.28$ (a) and $t=11.6$ (b). Other parameters are the same as in Fig. 4.

$$B'=|\beta_3|E^{-2}B,\quad C'=|\beta_4|E^{-2}C.$$

System (12) now turns into (primes are omitted)

$$\partial_T A=\beta_1\partial_Z^2A+\alpha_2RA-\alpha_3A\partial_ZB+\alpha_4A\partial_ZC,$$

$$\partial_T B=\partial_Z^2B-\partial_Z|A|^2, \tag{13}$$

$$\partial_T C=\tau\partial_Z^2C-\tau\alpha_6\partial_Z|A|^2,$$

where the coefficients $\alpha_2=\beta_{2R}E^{-2}$, $\alpha_3=\beta_3/|\beta_3|$, $\alpha_4=\beta_4/|\beta_4|$, and $\alpha_6=\beta_6|\beta_4|/(\beta_5|\beta_3|)$.

The system (13) was solved numerically by explicit and Dufort-Frankel finite-difference schemes. The boundary conditions were zero, and initial conditions were sinusoidal for dependent variables. For numerical experiments we chose the parameters $\sigma=7$, $K=\pi$, and $E=0.05$ and two values of the Lewis number: $\tau=1/10$ and $\tau=1/81$. The governing parameter $R$ had range from 0.1 to 50. The number of vertical nodes $n$ was varied from 256 to 2048 to resolve the microstructure.

In the cases when $R\gtrsim8$ the initial state was destroyed after some time by a multiple Eckhaus instability (the birth of convective cells [16]) and was followed by the solution of diffusive chaos type, with strong space-time irregularity [17]. In this case the mean profiles of temperature and salinity become perturbed so that the layer of inversion splits into 10–30 small layers (see Fig. 4 for $\tau=1/81$ and time $t=9.28$ h). Buoyancy frequency (Fig. 5) becomes very irregular, and all the fine structure slowly changes with time.

TABLE I. Parameter estimations for inversion of thermohaline staircase. For all cases $T=15$ °C; $S=36\%$; $\sigma=7$; $\tau=1/81$; $t_0$, diffusive time scale; $t_0'$, main time scale; $N_0=(1-\tau)/(1+\sigma)$, limit of critical buoyancy frequency at $\varepsilon=0$; and $400t_0'$, time of the establishment of diffusive chaos in the inversion.

| Parameter | 1 | 2 | 3 |
|---|---|---|---|
| $h$ [cm] | 400.0 | 250.0 | 100.0 |
| $(T_+-T_-)$ [°C] | 1.0 | 0.1 | 0.1 |
| $(S_+-S_-)$ [%] | 0.33 | 0.033 | 0.033 |
| $R_T$ | $9.5\times10^{11}$ | $2.3\times10^{10}$ | $1.48\times10^{9}$ |
| $R_S$ | $1.08\times10^{12}$ | $2.6\times10^{10}$ | $1.69\times10^{9}$ |
| $\varepsilon$ | 0.0006 | 0.0015 | 0.003 |
| $\delta$ | 0.012 | 0.03 | 0.06 |
| $l_c$ [cm] | 4.7 | 7.7 | 6.0 |
| $t_0$ [sec] | $1.12\times10^{8}$ | $4.38\times10^{7}$ | $7.0\times10^{6}$ |
| $t_0'$ [min] | 0.68 | 1.7 | 1.07 |
| $N_0$ [cyc/h] | 4.95 | 1.97 | 3.13 |
| $400t_0'$ [h] | 4.5 | 11.4 | 7.14 |

## IV. CONCLUSION

In this paper we have developed a mathematical model describing the formation of vertical convective patterns in two-dimensional double-diffusive convection in a limit of high Hopf frequency (large Rayleigh numbers) for an infinite horizontal layer. A physical system corresponding to this model is the formation of the fine structure of hydrophysical fields in the inversion layer of a thermohaline staircase. Some typical parameters of such inversions are presented in Table I. It is known [18] that parameters of stratification in the inversions are often near the onset of convection. Also the vertical microstructure (usually step like) is often observed in the inversions along with small-scale turbulence [19]. The results of this work are in qualitative agreement with these observations. For a detailed discussion of the experimental data see [5] and references therein. Although a detailed comparison with the observed data is out of the scope of this paper, we should note a few items of the qualitative agreement.

(i) The developed model predicts that fine structure should exist in a given system for a wide range of parameters with typical time of pattern formation of about a few hours.

(ii) Fine structure has a very irregular shape slowly changing with time in accordance with solution of the *ABC* system of diffusive chaos type. Mean profiles of temperature and salinity become perturbed so that the layer of inversion splits into 10–30 thin layers.

(iii) The buoyancy frequency structure has peak emissions with amplitude and width in reasonable agreement with the observational data [19] in the cases when temperature and salinity differences per layer are relatively small, as shown in Table I.

Of course, the developed model based on the *ABC* system of amplitude equations has its limitation of a weakly nonlinear approximation, which hardly allows to get a “full-fledged” steplike vertical structure of density, as noted in [6]. This defect can be partially overcome by regarding amplitude equations arising at higher orders of small parameters in the multiple-scale method. But this is a matter of further work, along with a more detailed comparison of predicted fine structure parameters with experimental and observational data.

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